Exercise 6E

1 a $\dfrac{3x+5}{(x+1)(x+2)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+2}$

$\Rightarrow 3x+5 \equiv A(x+2)+B(x+1)$

$x=-1 \Rightarrow 2 = A$

$x=-2 \Rightarrow -1 = -B \Rightarrow B = 1$

$\therefore \displaystyle\int \frac{3x+5}{(x+1)(x+2)}\,\mathrm{d}x = \int\left(\frac{2}{x+1}+\frac{1}{x+2}\right)\mathrm{d}x$

$= 2\ln|x+1| + \ln|x+2| + c$

$= \ln\left(|x+1|^2\right) + \ln|x+2| + c$

$= \ln\left|(x+1)^2(x+2)\right| + c$

b $\dfrac{3x-1}{(2x+1)(x-2)} \equiv \dfrac{A}{2x+1} + \dfrac{B}{x-2}$

$\Rightarrow 3x-1 \equiv A(x-2)+B(2x+1)$

$x=2 \Rightarrow 5 = 5B \Rightarrow B = 1$

$x=-\frac{1}{2} \Rightarrow -\frac{5}{2} = -\frac{5}{2}A \Rightarrow A = 1$

$\therefore \displaystyle\int \frac{3x-1}{(2x+1)(x-2)}\,\mathrm{d}x = \int\left(\frac{1}{2x+1}+\frac{1}{x-2}\right)\mathrm{d}x$

$= \frac{1}{2}\ln|2x+1| + \ln|x-2| + c$

$= \ln\left|(x-2)\sqrt{2x+1}\right| + c$

c $\dfrac{2x-6}{(x+3)(x-1)} \equiv \dfrac{A}{x+3} + \dfrac{B}{x-1}$

$\Rightarrow 2x-6 \equiv A(x-1)+B(x+3)$

$x=1 \Rightarrow -4 = 4B \Rightarrow B = -1$

$x=-3 \Rightarrow -12 = -4A \Rightarrow A = 3$

$\therefore \displaystyle\int \frac{2x-6}{(x+3)(x-1)}\,\mathrm{d}x = \int\left(\frac{3}{x+3}-\frac{1}{x-1}\right)\mathrm{d}x$

$= 3\ln|x+3| - \ln|x-1| + c$

$= \ln\left|\dfrac{(x+3)^3}{x-1}\right| + c$

d $\dfrac{3}{(2+x)(1-x)} \equiv \dfrac{A}{(2+x)} + \dfrac{B}{1-x}$

$\Rightarrow 3 \equiv A(1-x)+B(2+x)$

$x=1 \Rightarrow 3 = 3B \Rightarrow B = 1$

$x=-2 \Rightarrow 3 = 3A \Rightarrow A = 1$

$\therefore \displaystyle\int \frac{3}{(2+x)(1-x)}\,\mathrm{d}x = \int\left(\frac{1}{(2+x)}+\frac{1}{1-x}\right)\mathrm{d}x$

$= \ln|2+x| - \ln|1-x| + c$

$= \ln\left|\dfrac{2+x}{1-x}\right| + c$

2 a $\dfrac{2(x^2+3x-1)}{(x+1)(2x-1)} \equiv 1 + \dfrac{A}{x+1} + \dfrac{B}{2x-1}$

$\Rightarrow 2x^2+6x-2 \equiv (x+1)(2x-1)$
$\quad + A(2x-1)+B(x+1)$

$x=-1 \Rightarrow -6 = -3A \Rightarrow A = 2$

$x=\frac{1}{2} \Rightarrow \frac{3}{2} = \frac{3}{2}B \Rightarrow B = 1$

$\therefore \displaystyle\int \frac{2(x^2+3x-1)}{(x+1)(2x-1)}\,\mathrm{d}x = \int\left(1+\frac{2}{x+1}+\frac{1}{2x-1}\right)\mathrm{d}x$

$= x + 2\ln|x+1| + \frac{1}{2}\ln|2x-1| + c$

$= x + \ln\left|(x+1)^2\sqrt{2x-1}\right| + c$

2 b $\dfrac{x^3+2x^2+2}{x(x+1)} \Rightarrow$

$$\begin{array}{r} x+1 \\ x^2+x\overline{)x^3+2x^2+2} \\ \underline{x^3+x^2} \\ x^2+2 \\ \underline{x^2+x} \\ 2-x \end{array}$$

$$\frac{x^3+2x^2+2}{x(x+1)} \equiv x+1+\frac{2-x}{x(x+1)}$$

$$\equiv x+1+\frac{A}{x}+\frac{B}{x+1}$$

$$\Rightarrow x^3+2x^2+2 \equiv (x+1)x(x+1) + A(x+1)+Bx$$

$x=0 \Rightarrow 2=A \Rightarrow A=2$

$x=-1 \Rightarrow 3=-B \Rightarrow B=-3$

$$\therefore \int \frac{x^3+2x^2+2}{x(x+1)}\,dx = \int\left(x+1+\frac{2}{x}-\frac{3}{x+1}\right)dx$$

$$=\frac{x^2}{2}+x+2\ln|x|-3\ln|x+1|+c$$

$$=\frac{x^2}{2}+x+\ln\left|\frac{x^2}{(x+1)^3}\right|+c$$

c $\dfrac{x^2}{x^2-4} \equiv 1+\dfrac{A}{x-2}+\dfrac{B}{x+2}$

$$\Rightarrow x^2 \equiv (x-2)(x+2)+A(x+2)+B(x-2)$$

$x=2 \Rightarrow 4=4A \Rightarrow A=1$

$x=-2 \Rightarrow 4=-4B \Rightarrow B=-1$

$$\therefore \int \frac{x^2}{x^2-4}\,dx = \int\left(1+\frac{1}{x-2}-\frac{1}{x+2}\right)dx$$

$$=x+\ln|x-2|-\ln|x+2|+c$$

$$=x+\ln\left|\frac{x-2}{x+2}\right|+c$$

d $\dfrac{x^2+x+2}{3-2x-x^2} \equiv \dfrac{x^2+x+2}{(3+x)(1-x)}$

$$\equiv -1+\frac{A}{3+x}+\frac{B}{1-x}$$

$$\Rightarrow x^2+x+2 \equiv -1(3+x)(1-x) + A(1-x)+B(3+x)$$

$x=1 \Rightarrow 4=4B \Rightarrow B=1$

$x=-3 \Rightarrow 8=4A \Rightarrow A=2$

$$\therefore \int \frac{x^2+x+2}{3-2x-x^2}\,dx = \int\left(-1+\frac{2}{3+x}+\frac{1}{1-x}\right)dx$$

$$=-x+2\ln|3+x|-\ln|1-x|+c$$

$$=-x+\ln\left|\frac{(3+x)^2}{1-x}\right|+c$$

3 a $f(x)=\dfrac{4}{(2x+1)(1-2x)}$

$$\frac{4}{(2x+1)(1-2x)}=\frac{A}{2x+1}+\frac{B}{1-2x}$$

$4=A(1-2x)+B(2x+1)$

Let $x=\frac{1}{2}: 4=2B \Rightarrow B=2$

Let $x=-\frac{1}{2}: 4=2A \Rightarrow A=2$

b $\displaystyle\int f(x)\,dx=\int\left(\frac{2}{(2x+1)}+\frac{2}{(1-2x)}\right)dx$

$$=\ln|2x+1|-\ln|1-2x|+c$$

$$=\ln\left|\frac{2x+1}{1-2x}\right|+c$$

c $\displaystyle\int_1^2 f(x)\,dx=\left[\ln\left|\frac{2x+1}{1-2x}\right|\right]_1^2$

$$=\ln\frac{5}{3}-\ln 3=\ln\frac{5}{9}$$

$$k=\frac{5}{9}$$

4 a $f(x)=\dfrac{17-5x}{(3+2x)(2-x)^2}$

$$\frac{17-5x}{(3+2x)(2-x)^2}=\frac{A}{3+2x}+\frac{B}{(2-x)^2}+\frac{C}{2-x}$$

$17-5x=A(2-x)^2+B(3+2x)+C(3+2x)(2-x)$

Let $x=2:7=7B\Rightarrow B=1$

Let $x=-\frac{3}{2}:17+\frac{15}{2}=\frac{49}{4}A\Rightarrow A=2$

Let $x=0:17=4A+3B+6C$

$\Rightarrow 17=8+3+6C\Rightarrow C=1$

$$f(x)=\frac{2}{3+2x}+\frac{1}{(2-x)^2}+\frac{1}{2-x}$$

b $\displaystyle\int_0^1\left(\frac{2}{3+2x}+\frac{1}{2-x}+\frac{1}{(2-x)^2}\right)dx$

$=\left[\ln|3+2x|-\ln|2-x|+\dfrac{1}{(2-x)}\right]_0^1$

$=(\ln 5-\ln 1+1)-(\ln 3-\ln 2+\frac{1}{2})$

$=\frac{1}{2}+\ln\frac{10}{3}$

5 a $f(x)=\dfrac{9x^2+4}{9x^2-4}$

Dividing gives:

$f(x)=1+\dfrac{8}{9x^2-4}$

$=1+\dfrac{8}{(3x+2)(3x-2)}$

$$\frac{8}{(3x+2)(3x-2)}=\frac{B}{3x-2}+\frac{C}{3x+2}$$

$8=B(3x+2)+C(3x-2)$

Let $x=-\frac{2}{3}:8=-4C\Rightarrow C=-2$

Let $x=\frac{2}{3}:8=4B\Rightarrow B=2$

$A=1, B=2, C=-2$

b $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}}\left(1+\frac{2}{3x-2}-\frac{2}{3x+2}\right)dx$

$=\left[x+\frac{2}{3}\ln|3x-2|-\frac{2}{3}\ln|3x+2|\right]_{-\frac{1}{3}}^{\frac{1}{3}}$

$=(\frac{1}{3}-\frac{2}{3}\ln 3)-(-\frac{1}{3}+\frac{2}{3}\ln 3)$

$=\frac{2}{3}-\frac{4}{3}\ln 3$

$a=\frac{2}{3}, b=-\frac{4}{3}, c=3$

6 a $f(x)=\dfrac{6+3x-x^2}{x^3+2x^2}=\dfrac{6+3x-x^2}{x^2(x+2)}$

$$\frac{6+3x-x^2}{x^2(x+2)}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{x+2}$$

$6+3x-x^2=A(x+2)+Bx(x+2)+Cx^2$

Let $x=0:6=2A\Rightarrow A=3$

Let $x=-2:-4=4C\Rightarrow C=-1$

Let $x=1:8=3A+3B+C\Rightarrow B=0$

$$f(x)=\frac{3}{x^2}-\frac{1}{x+2}$$

b $\displaystyle\int_2^4\frac{6+3x-x^2}{x^3+2x^2}\,dx$

$=\displaystyle\int_2^4\left(\frac{3}{x^2}-\frac{1}{x+2}\right)dx$

$=\left[-\dfrac{3}{x}-\ln|x+2|\right]_2^4$

$=\left(-\dfrac{3}{4}-\ln 6\right)-\left(-\dfrac{3}{2}-\ln 4\right)$

$=\dfrac{3}{4}+\ln\dfrac{2}{3}$

$a=\frac{3}{4}, b=\frac{2}{3}$

7 a Let $f(x)=\dfrac{32x^2+4}{(4x+1)(4x-1)}$

Dividing:

$$\frac{32x^2+4}{(4x+1)(4x-1)}=2+\frac{6}{(4x+1)(4x-1)}$$

$\Rightarrow A=2$

$$\frac{6}{(4x+1)(4x-1)}=\frac{B}{4x+1}+\frac{C}{4x-1}$$

$6=B(4x-1)+C(4x+1)$

Let $x=\frac{1}{4}:6=2C\Rightarrow C=3$

Let $x=-\frac{1}{4}:6=-2B\Rightarrow B=-3$

$$f(x)=2-\frac{3}{4x+1}+\frac{3}{4x-1}$$

b $\displaystyle\int_1^2 f(x)\,dx=\int_1^2\left(2-\frac{3}{4x+1}+\frac{3}{4x-1}\right)dx$

$=\left[2x-\frac{3}{4}\ln|4x+1|+\frac{3}{4}\ln|4x-1|\right]_1^2$

$=(4-\frac{3}{4}\ln 9+\frac{3}{4}\ln 7)-(2-\frac{3}{4}\ln 5+\frac{3}{4}\ln 3)$

$=2+\frac{3}{4}(-\ln 9+\ln 7+\ln 5-\ln 3)$

$=2+\frac{3}{4}\ln\frac{35}{27}$, so $k=\frac{3}{4}, m=\frac{35}{27}$